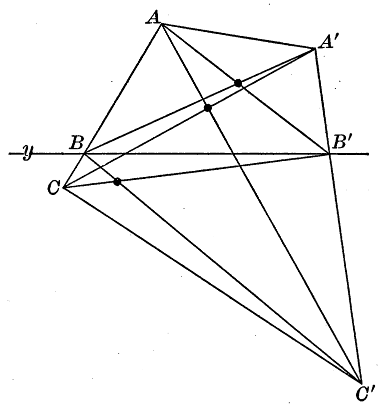
The intersections of the opposite sides are collinear, by the theorem of Pappus, that is, if a hexagon $AB'CA'BC'$ has its vertices of odd order on one straight line, and its vertices of even order on another straight line, then the three pairs of opposite sides, AB' and $A'B$, $B'C$ and BC', CA' and $C'A$, meet in three points lying on another straight line.



But the intersections of the opposite sides of this hexagon are the intersections of the diagonals of the three quadrilaterals.

Hence, the intersections of the diagonals of any three quadrilaterals, two of which are formed by cutting the other one by a straight line, are collinear.

Also solved by ANNA MULLIKIN.

A solution of this problem appeared in the January issue of the MONTHLY, but we publish this solution as it presents an entirely different method of attack. EDITORS.

452. Proposed by NATHAN ALTSHILLER, University of Washington.

Through a given point a secant is drawn that meets three given concurrent lines in the points A, B, C, respectively. Determine the position of the secant by the condition $AB/BC = K$, K being given.

SOLUTION BY MRS. ELIZABETH B. DAVIS, U. S. Naval Observatory.

Let OA', OB' and OC' be three given concurrent lines, and P a given point. Let it be required to draw through P a secant meeting OA', OB' and OC' respectively in points A, B, and C, such that $AB/BC = K$, K being given.

Join P and O, and through P draw any transversal $R'P$, meeting the four lines of the pencil $O - A'B'C'P$ in D, E, F, and P, respectively.

On OP take H and G so that

$$GP : HP = DE : EF. \qquad (1)$$

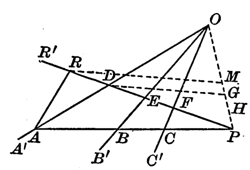
Also, on OP take M, so that

$$MP : HP = K. \qquad (2)$$

Join DG, and through M draw RM parallel to DG, meeting $R'P$ in R.

Draw RA parallel to OC', meeting OA' in A. Join AP, then AP is the transversal required. For, dividing (1) by (2), we have

$$GP : MP = \frac{DE}{EF} : K.$$



Since, $\triangle$'s PDG and PRM are similar,

$$GP : MP = DP : RP.$$

Hence

$$DP : RP = \frac{DE}{EF} : K. \qquad (3)$$

Dividing the first ratio of (3) by FP,

$$\frac{DP}{FP} : \frac{RP}{FP} = \frac{DE}{EF} : K. \qquad (4)$$

Since $\triangle$'s RAP and FCP are similar,

$$\frac{RP}{FP} = \frac{AP}{CP}.$$

Substituting this value of RP/FP in (4), we have

$$\frac{DP}{FP} : \frac{AP}{CP} = \frac{DE}{EF} : K,$$

or

$$\frac{DP}{FP} : \frac{DF}{EF} = \frac{AP}{CP} : K. \qquad (5)$$

But the anharmonic ratio of all transversals cutting any given pencil of four lines, as $O-A'B'C'P$, is a constant. Hence,

$$\frac{DP}{FP} : \frac{DE}{EF} = \frac{AP}{CP} : \frac{AB}{BC}. \qquad (6)$$

From (5) and (6), $AB/BC = K$. Hence AP is the transversal required.

Also solved by the PROPOSER.

453. Proposed by CLIFFORD N. MILLS, Brookings, S. D.

Prove geometrically the formulæ for $\sin 2\beta$, $\cos 2\beta$, $\sin 3\beta$, $\cos 3\beta$.

SOLUTION BY THE PROPOSER.

(1) $\sin 2\beta = 2 \sin \beta \cos \beta$.

Inscribe in a circle any triangle, ABC, with the angle at B equal to 2β (Fig. 1). Draw AM through the center of the circle; BK, the bisector of the angle B; OK, the radius of the circle; AK, KM, and KC, K being on the circumference of the circle.

Then

$$\sin 2\beta = \frac{AC}{2r}.$$

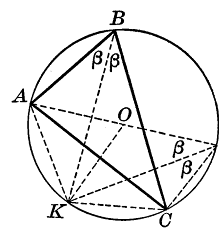


Fig. 1.

$\triangle$ AKC and KOM are similar isosceles triangles. Hence

$$\frac{AC}{AK} = \frac{KM}{r} \quad \text{or} \quad AC = \frac{AK \cdot KM}{r}.$$

Hence,

$$\frac{AC}{2r} = \frac{AK \cdot KM}{2r^2} = \frac{2AK \cdot KM}{4r^2}.$$

But

$$\frac{AK}{2r} = \sin \beta \quad \text{and} \quad \frac{KM}{2r} = \cos \beta.$$

Hence $\sin 2\beta = 2 \sin \beta \cos \beta$.

(2) $\cos 2\beta = \cos^2 \beta - \sin^2 \beta$.

Using the same Fig. 1, we get

$$\cos 2\beta = \frac{MC}{2r} = \sqrt{\frac{4r^2 - \overline{AC}^2}{2r}} = \sqrt{1 - \frac{\overline{AC}^2}{4r^2}}.$$

But

$$AC = (AK \cdot KM)/r.$$

Hence,

$$\cos 2\beta = \sqrt{1 - \frac{4\overline{AK}^2 \cdot \overline{KM}^2}{16r^4}}.$$

Substituting the values of AK and KM, as previously found, we have

$$\cos 2\beta = \sqrt{1 - 4 \sin^2 \beta \cos^2 \beta} = \sqrt{(\sin^2 \beta + \cos^2 \beta)^2 - 4 \sin^2 \beta \cos^2 \beta} = \cos^2 \beta - \sin^2 \beta.$$

(3) $\sin 3\beta = 3 \sin \beta - 4 \sin^3 \beta$.